

# More Proofs of Divergence of the Harmonic Series

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In an earlier article, Kifowit and Stamps [5] summarized a number of elementary proofs of divergence of the harmonic series. For a variety of reasons, some very nice proofs never made it into the final draft of that article. With this in mind, the collection of divergence proofs continues here. This informal note is a work in progress<sup>1</sup>. On occasion, more proofs will be added. Accessibility to first-year calculus students is a common thread that will continue (usually) to connect the proofs.

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The harmonic series,  $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$ , diverges.

PROOF 21: (A geometric series proof)

Choose a positive integer  $k$ .

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \overbrace{\left( \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k+1} \right)}^{k \text{ terms}} + \overbrace{\left( \frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{k^2+k+1} \right)}^{k^2 \text{ terms}} \\ &\quad + \overbrace{\left( \frac{1}{k^2+k+2} + \frac{1}{k^2+k+3} + \dots + \frac{1}{k^3+k^2+k+1} \right)}^{k^3 \text{ terms}} + \dots \\ &> 1 + \frac{k}{k+1} + \frac{k^2}{k^2+k+1} + \frac{k^3}{k^3+k^2+k+1} + \dots \\ &> 1 + \left( \frac{k}{k+1} \right) + \left( \frac{k}{k+1} \right)^2 + \left( \frac{k}{k+1} \right)^3 + \dots \\ &= \frac{1}{1 - \frac{k}{k+1}} = k+1 \end{aligned}$$

Since this is true for any positive integer  $k$ , the harmonic series must diverge.

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<sup>1</sup>Last updated on February 29, 2008

PROOF 22:

The following proof was given by Fearnough [3] and later by Havil [4]. After substituting  $u = e^x$ , this proof is equivalent to Proof 10 of [5].

$$\begin{aligned}\int_{-\infty}^0 \frac{e^x}{1 - e^x} dx &= \int_{-\infty}^0 e^x(1 - e^x)^{-1} dx \\ &= \int_{-\infty}^0 e^x(1 + e^x + e^{2x} + e^{3x} + \dots) dx \\ &= \int_{-\infty}^0 (e^x + e^{2x} + e^{3x} + \dots) dx \\ &= \left[ e^x + \frac{1}{2}e^{2x} + \frac{1}{3}e^{3x} + \dots \right]_{-\infty}^0 \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \dots \\ &= [-\ln(1 - e^x)]_{-\infty}^0 = \infty\end{aligned}$$

PROOF 23: (A telescoping series proof)

This proof was given by Bradley [2].

We begin with the inequality  $x \geq \ln(1 + x)$ , which holds for all  $x > -1$ . From this, it follows that

$$\frac{1}{k} \geq \ln\left(1 + \frac{1}{k}\right) = \ln(k + 1) - \ln(k)$$

for any positive integer  $k$ . Now we have

$$\begin{aligned}H_n &= \sum_{k=1}^n \frac{1}{k} \\ &\geq \sum_{k=1}^n \ln\left(1 + \frac{1}{k}\right) = \sum_{k=1}^n \ln\left(\frac{k+1}{k}\right) \\ &= (\ln(n+1) - \ln(n)) + (\ln(n) - \ln(n-1)) + \dots + ((\ln(2) - \ln(1))) \\ &= \ln(n+1).\end{aligned}$$

Therefore  $\{H_n\}$  is unbounded, and the harmonic series diverges.

PROOF 24: (A limit comparison proof)

In the last proof, the harmonic series was directly compared to the divergent series  $\sum_{k=1}^{\infty} \ln\left(1 + \frac{1}{k}\right)$ . The use of the inequality  $x \geq \ln(1 + x)$  can be avoided by using limit comparison:

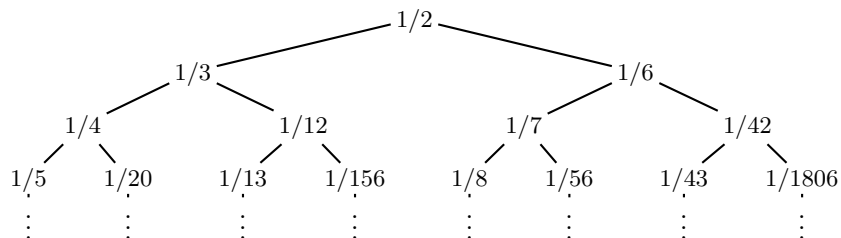
Since

$$\lim_{x \rightarrow \infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{-\frac{1}{x^2}}{\left(1 + \frac{1}{x}\right)\left(-\frac{1}{x^2}\right)} = 1,$$

the harmonic series diverges by limit comparison.

PROOF 25:

In a very interesting proof of the Egyptian fraction theorem, Owings [7] showed that no number appears more than once in any single row of the following tree.



The elements of each row have a sum of  $1/2$ , and there are infinitely rows with no elements in common. (For example, one could, starting with row 1, find the maximum denominator in the row, and then jump to that row.) It follows that the harmonic series diverges.

PROOF 26:

This proof is actually a pair of very similar proofs. They are closely related to a number of other proofs, but most notably to Proof 4 of [5]. In these proofs,  $H_n$  denotes the  $n$ th partial sum of the harmonic series:

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \quad n = 1, 2, 3, \dots$$

Proof (A): First notice that

$$H_n + H_{2n} = 2H_n + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \geq 2H_n + \frac{n}{2n},$$

so that when all is said and done, we have

$$H_n + H_{2n} \geq 2H_n + \frac{1}{2}.$$

Now suppose the harmonic series converges with sum  $S$ .

$$\begin{aligned} 2S &= \lim_{n \rightarrow \infty} H_n + \lim_{n \rightarrow \infty} H_{2n} \\ &= \lim_{n \rightarrow \infty} (H_n + H_{2n}) \\ &\geq \lim_{n \rightarrow \infty} \left( 2H_n + \frac{1}{2} \right) \\ &= 2S + \frac{1}{2} \end{aligned}$$

The contradiction  $2S \geq 2S + \frac{1}{2}$  concludes the proof.

Proof (B): This proof was given by Ward [9].

$$H_{2n} - H_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \geq \frac{n}{2n} = \frac{1}{2}$$

Suppose the harmonic series converges.

$$\begin{aligned}
 0 &= \lim_{n \rightarrow \infty} H_{2n} - \lim_{n \rightarrow \infty} H_n \\
 &= \lim_{n \rightarrow \infty} (H_{2n} - H_n) \\
 &\geq \lim_{n \rightarrow \infty} \frac{1}{2} = \frac{1}{2}
 \end{aligned}$$

The contradiction  $0 \geq \frac{1}{2}$  concludes the proof.

PROOF 27:

This proposition follows immediately from the harmonic mean/arithmetic mean inequality, but an alternate proof is given here.

*Proposition:* For any natural number  $k$ ,  $\frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{3k} > 1$ .

Proof:

$$\begin{aligned}
 \exp\left(\frac{1}{k} + \frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{3k}\right) &= e^{1/k} \cdot e^{1/(k+1)} \cdot e^{1/(k+2)} \cdots e^{1/(3k)} \\
 &> \left(1 + \frac{1}{k}\right) \cdot \left(1 + \frac{1}{k+1}\right) \cdot \left(1 + \frac{1}{k+2}\right) \cdots \left(1 + \frac{1}{3k}\right) \\
 &= \left(\frac{k+1}{k}\right) \cdot \left(\frac{k+2}{k+1}\right) \cdot \left(\frac{k+3}{k+2}\right) \cdots \left(\frac{3k+1}{3k}\right) \\
 &= \frac{3k+1}{k} > 3.
 \end{aligned}$$

(In the proposition, the denominator  $3k$  could be replaced by  $[ek]$ , but even this choice is not optimal. See [1].)

Based on this proposition, we have the following result:

$$\begin{aligned}
 \sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \left(\frac{1}{2} + \cdots + \frac{1}{6}\right) + \left(\frac{1}{7} + \cdots + \frac{1}{21}\right) + \left(\frac{1}{22} + \cdots + \frac{1}{66}\right) + \cdots \\
 &> 1 + 1 + 1 + 1 + \cdots
 \end{aligned}$$

PROOF 28: (A Fibonacci number proof)

The Fibonacci numbers are defined recursively as follows:

$$f_0 = 1, \quad f_1 = 1; \quad f_{n+1} = f_n + f_{n-1}, \quad n = 1, 2, 3, \dots$$

For example, the first ten are given by 1, 1, 2, 3, 5, 8, 13, 21, 34, 55. The sequence of Fibonacci numbers makes an appearance in a number of modern calculus textbooks (for instance, see [6] or [8]). Often, the limit

$$\lim_{n \rightarrow \infty} \frac{f_{n+1}}{f_n} = \phi = \frac{1 + \sqrt{5}}{2}$$

is proved or presented as an exercise. This limit plays an important role in the following divergence proof.

First notice that

$$\lim_{n \rightarrow \infty} \frac{f_{n-1}}{f_{n+1}} = \lim_{n \rightarrow \infty} \frac{f_{n+1} - f_n}{f_{n+1}} = \lim_{n \rightarrow \infty} \left( 1 - \frac{f_n}{f_{n+1}} \right) = 1 - \frac{1}{\phi} \approx 0.381966.$$

Now we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \left( \frac{1}{4} + \frac{1}{5} \right) + \left( \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right) \\ &\quad + \left( \frac{1}{9} + \dots + \frac{1}{13} \right) + \left( \frac{1}{14} + \dots + \frac{1}{21} \right) + \dots \\ &\geq 1 + \frac{1}{2} + \frac{1}{3} + \frac{2}{5} + \frac{3}{8} + \frac{5}{13} + \frac{8}{21} + \dots \\ &= 1 + \sum_{n=1}^{\infty} \frac{f_{n-1}}{f_{n+1}} \end{aligned}$$

Since  $\lim_{n \rightarrow \infty} \frac{f_{n-1}}{f_{n+1}} \neq 0$ , this last series diverges. It follows that the harmonic series diverges.

PROOF 29:

This proof is very much like Proof 2 of [5]. First notice that since the sequence

$$\frac{11}{10}, \frac{111}{100}, \frac{1111}{1000}, \frac{11111}{10000}, \dots$$

increases and converges to 10/9, the sequence

$$\frac{10}{11}, \frac{100}{111}, \frac{1000}{1111}, \frac{10000}{11111}, \dots$$

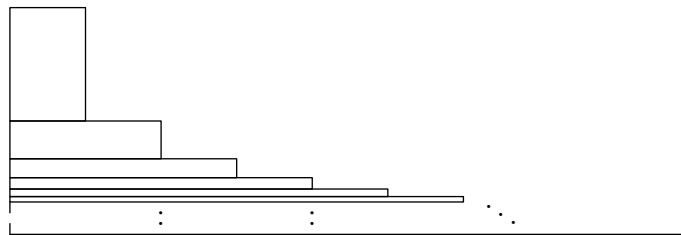
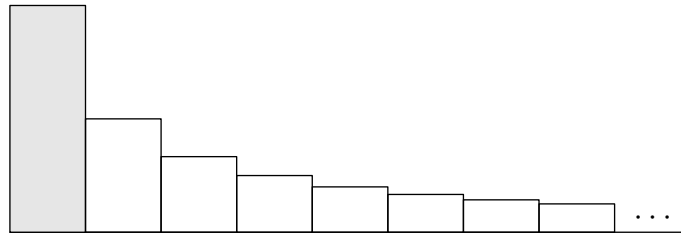
decreases and converges to 9/10. With this in mind, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \overbrace{\left( \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{11} \right)}^{10 \text{ terms}} + \overbrace{\left( \frac{1}{12} + \frac{1}{13} + \dots + \frac{1}{111} \right)}^{100 \text{ terms}} \\ &\quad + \overbrace{\left( \frac{1}{112} + \frac{1}{113} + \dots + \frac{1}{1111} \right)}^{1000 \text{ terms}} + \dots \\ &> 1 + \frac{10}{11} + \frac{100}{1111} + \frac{1000}{11111} + \dots \\ &> 1 + \frac{9}{10} + \frac{9}{10} + \frac{9}{10} + \dots \end{aligned}$$

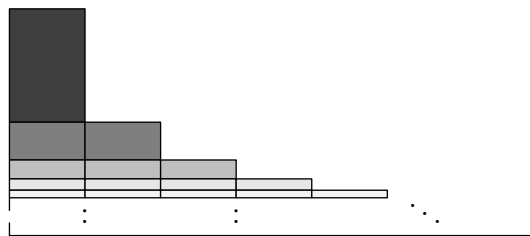
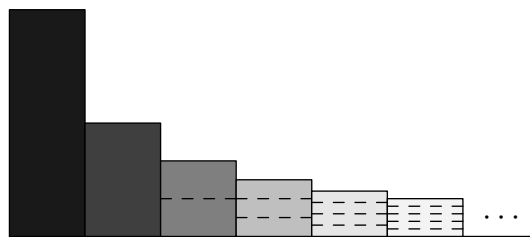
PROOF 30:

The following visual proofs each show that by carefully rearranging terms, the harmonic series can be made greater than itself.

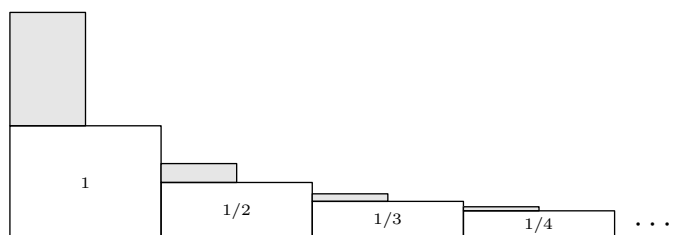
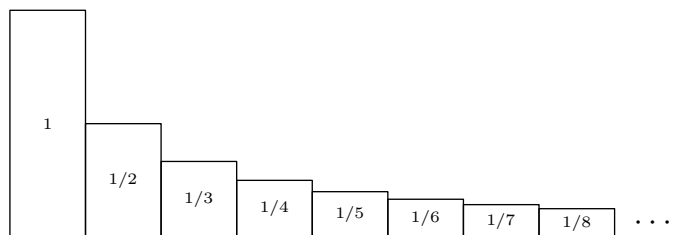
Proof (A): This proof without words was posted on *The Everything Seminar* (Harmonic Digression, <http://cornellmath.wordpress.com/2007/07/12/harmonic-digression/>).



Proof (B): This visual proof is due to Jim Belk. It was posted on *The Everthing Seminar* as a follow-up to the previous proof. Belk's proof is a visualization of Johann Bernoulli's proof (see Proof 13 of [5]).



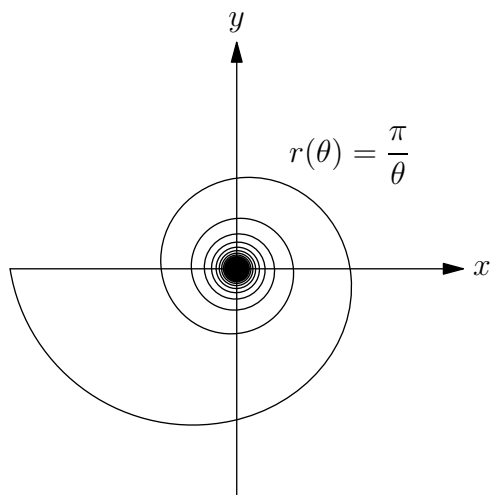
Proof (C): This proof is a visual representation of Proof 6 (and Proof 7) of [5].



PROOF 31:

Here is another proof in which  $\sum 1/k$  and  $\int 1/x dx$  are compared. Unlike its related proofs (e.g. Proof 9 of [5]), this one focuses on arc length.

The graph shown here is that of the polar function  $r(\theta) = \pi/\theta$  on  $[\pi, \infty)$ .



The total arc length is unbounded:

$$\text{Arc Length} = \int_{\pi}^{\infty} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \geq \int_{\pi}^{\infty} \sqrt{r^2} d\theta = \int_{\pi}^{\infty} r d\theta = \int_{\pi}^{\infty} \frac{\pi}{\theta} d\theta = \pi \ln \theta \Big|_{\pi}^{\infty} = \infty$$

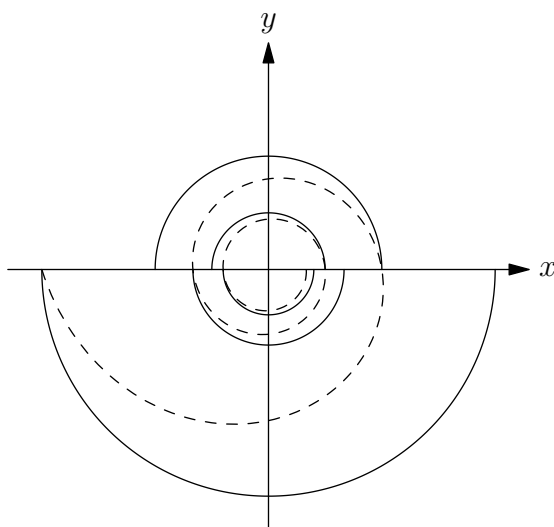
Now let the polar function  $\rho$  be defined by

$$\rho(\theta) = \begin{cases} 1, & \pi \leq \theta < 2\pi \\ 1/2, & 2\pi \leq \theta < 3\pi \\ 1/3, & 3\pi \leq \theta < 4\pi \\ \vdots & \vdots \\ 1/n, & n\pi \leq \theta < (n+1)\pi \\ \vdots & \vdots \end{cases}$$

The graph of  $\rho$  is made up of semi-circular arcs, the  $n$ th arc having radius  $1/n$ . The total arc length of the graph of  $\rho$  is

$$\pi + \frac{\pi}{2} + \frac{\pi}{3} + \frac{\pi}{4} + \cdots = \pi \left( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \right).$$

The graphs of  $r$  (dashed) and  $\rho$  (solid) on  $[\pi, 6\pi]$  are shown below.



By comparing the graphs of the two functions over intervals of the form  $[n\pi, (n+1)\pi]$ , it is easy to see that the graph of  $\rho$  must be “longer” than the graph of  $r$ . It follows that

$$\pi \left( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \right)$$

must be unbounded.

### PROOF 32:

The divergence of the harmonic series follows immediately from the Cauchy Condensation Test:

*Suppose  $\{a_n\}$  is a non-increasing sequence with positive terms. Then  $\sum_{n=1}^{\infty} a_n$  converges if and*

*only if  $\sum_{k=0}^{\infty} 2^k a_{2^k}$  converges.*

## References

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- [2] D. M. BRADLEY, *A note on the divergence of the harmonic series*, American Mathematical Monthly, 107 (2000), p. 651.
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